Divide-and-Conquer Fusion: Methods for unifying distributed analyses

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Warwick Statistics

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Outline

Fusion Problem

- Target fusion density:

$$f(\boldsymbol{x}) \propto \prod_{c=1}^{C} f_c(\boldsymbol{x})$$

  where each *sub-posterior*, $f_c(\boldsymbol{x})$, is a density representing one of the C distributed inferences we wish to unify

  - No general analytical approach
  - Monte Carlo: assume we can sample $\boldsymbol{x}^{(c)} \sim f_c(\boldsymbol{x})$
- Applications:
  - Expert elicitation: combining views of multiple experts
  - Big Data (by construction)
    - Partitioning large datasets to make them more manageable
  - Inference in privacy settings

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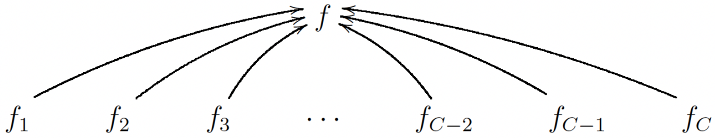
Fork-and-join

The fork-and-join approach:



Some Fork-and-Join methods

- Several fork-and-join methods have been developed (typically for Bayesian inference for large dataset applications):
  - Kernel density averaging (KDEMC) [Neiswanger et al., 2014]
  - Weierstrass sampler (WRS) [Wang and Dunson, 2013]
  - Consensus Monte Carlo (CMC) [Scott et al., 2016]
- Generally the recombination is inexact and involve approximations
  - CMC is exact if all sub-posteriors are Gaussian
  - All theory is asymptotic in the number of observations
- However, Monte Carlo Fusion [Dai et al., 2019] (and subsequently Bayesian Fusion [Dai et al., 2023]) is exact in the sense it targets the correct fusion density

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An extended target density

Proposition

Suppose that $p_c(\boldsymbol{y}|\boldsymbol{x}^{(c)})$ is the transition density of a stochastic process with stationary distribution $f_c^2(\boldsymbol{x})$. The $(C+1)d$-dimensional (fusion) density proportional to the integrable function

$$g\left(\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(C)}, \boldsymbol{y}\right) \propto \prod_{c=1}^{C}\left[f_c^2\left(\boldsymbol{x}^{(c)}\right) \cdot p_c\left(\boldsymbol{y}\middle|\boldsymbol{x}^{(c)}\right) \cdot \frac{1}{f_c(\boldsymbol{y})}\right]$$

admits the marginal density f for $\boldsymbol{y}$.

Main idea: If we can sample from g, then we can can obtain a draw from the fusion density ($\boldsymbol{y} \sim f$)

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An extended target density

- There are many possible choices for $p_c(\boldsymbol{y}|\boldsymbol{x}^{(c)})$
- Let $p_c(\boldsymbol{y}|\boldsymbol{x}^{(c)}) := p_{T,c}(\boldsymbol{y}|\boldsymbol{x}^{(c)})$, the transition density of the d-dimensional (double) Langevin (DL) diffusion processes $\boldsymbol{X}_t^{(c)}$ from $\boldsymbol{x}^{(c)}$ to $\boldsymbol{y}$ for $c = 1, \ldots, C$, for a pre-defined time $T > 0$ given by

$$\mathrm{d}\boldsymbol{X}_t^{(c)} = \nabla \log f_c \left(\boldsymbol{X}_t^{(c)}\right) \mathrm{d}t + \mathrm{d}\boldsymbol{W}_t^{(c)},$$

  (where $\boldsymbol{W}_t^{(c)}$ is d-dimensional Brownian motion and ∇ is the gradient operator over $\boldsymbol{x}$)
  - Has stationary distribution $f_c^2(\boldsymbol{x})$
  - Sample paths of DL diffusions can be simulated exactly using Path Space Rejection Sampling / Exact Algorithm methodology [Beskos et al., 2005, 2006; Pollock et al., 2016]

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Constructing a rejection sampler for g

- Extended target density:

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1. Simulate $\boldsymbol{x}^{(c)} \sim f_c(\boldsymbol{x})$ independently
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Rejection sampling - acceptance probability

- Acceptance probability:

$$\frac{g(\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(C)}, \boldsymbol{y})}{h(\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(C)}, \boldsymbol{y})} \propto \rho_0 \cdot \rho_1$$

where

$$\begin{cases} \rho_0 := e^{-\frac{C\sigma^2}{2T}}, & \sigma^2 = \frac{1}{C}\sum_{c=1}^{C} \left\| \boldsymbol{x}^{(c)} - \bar{\boldsymbol{x}} \right\|^2 \\ \rho_1 := \mathbb{E}_{\bar{\mathbb{W}}}\left(\prod_{c=1}^{C}\left[\exp\left\{-\int_0^T \left(\phi_c\left(\boldsymbol{X}_t^{(c)}\right) - \boldsymbol{\Phi}_c\right) \mathrm{d}t\right\}\right]\right) \end{cases}$$

where $\bar{\mathbb{W}}$ denotes the law of C independent Brownian bridges $\boldsymbol{X}_t^{(1)}, \ldots, \boldsymbol{X}_t^{(C)}$ with $\boldsymbol{X}_0^{(c)} = \boldsymbol{x}^{(c)}$ and $\boldsymbol{X}_T^{(c)} = \boldsymbol{y}$

- Trade-off with choice of T: as T increases, ρ_0 increases, but this results in ρ_1 to be small (might typically decrease exponentially with T)

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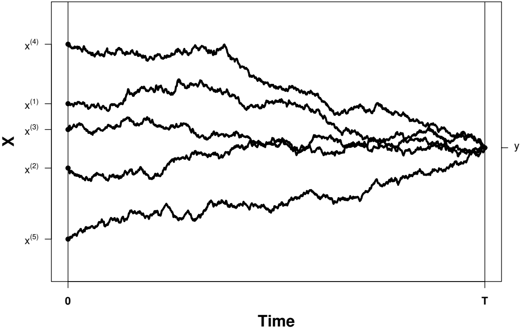
Monte Carlo Fusion - Interpretation

- Correct a simple average $\bar{x}$ of sub-posterior values to obtain samples for f
- The correction comes by simulating y from a Gaussian centred at $\bar{x}$, and accepting those samples with probability proportional to $\rho_0 \cdot \rho_1$



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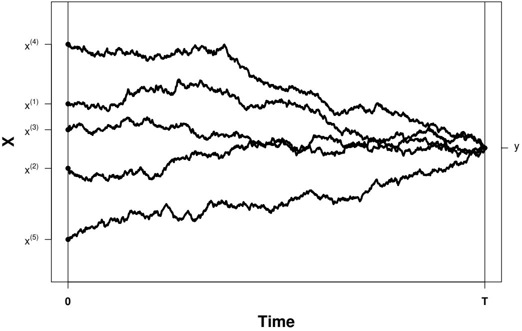

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Limitations of Monte Carlo Fusion

- Robustness: there is a lack of robustness when:
    - sub-posterior correlation increases
    - C increases
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- Aim: To construct a fusion algorithm / framework to alleviate some of these limitations (see Dai et al. [2023]; Chan et al. [2021, 2023] for full details)

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The Generalised Monte Carlo Fusion (GMCF) approach

Problem: Scalability with sub-posterior correlation

- Recall we have the flexibility to choose different p_c (transition density of stochastic process with f_c^2 invariant density)
- Now, we choose p_c to be the transition density of the d-dimensional (double) Langevin (DL) diffusion processes $\boldsymbol{X}_t^{(c)}$ with covariance matrix, $\boldsymbol{\Lambda}_c$ from $\boldsymbol{x}^{(c)}$ to $\boldsymbol{y}$ for $c = 1, \ldots, C$, over $[0, T]$ given by

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Constructing an importance sampler

- Switch to importance sampler for the extended target density $g(\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(C)}, \boldsymbol{y})$:
  - Rejection sampling can be wasteful
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$$\begin{cases} \rho_0 := \exp\left\{-\sum_{c=1}^{C} \frac{(\tilde{\boldsymbol{x}}-\boldsymbol{x}^{(c)})^\intercal \boldsymbol{\Lambda}_c^{-1}(\tilde{\boldsymbol{x}}-\boldsymbol{x}^{(c)})}{2T}\right\} \\ \rho_1 := \prod_{c=1}^{C} \mathbb{E}_{\mathbb{W}_{\boldsymbol{\Lambda}_c}}\left[\exp\left\{-\int_0^T \left(\phi_c\left(\boldsymbol{X}_t^{(c)}\right) - \boldsymbol{\Phi}_c\right) \mathrm{d}t\right\}\right] \end{cases}$$

where $\phi_c(\boldsymbol{x}) := \frac{1}{2}\left(\nabla \log f_c(\boldsymbol{x})^\intercal \boldsymbol{\Lambda}_c \nabla \log f_c(\boldsymbol{x}) + \mathrm{Tr}\left(\boldsymbol{\Lambda}_c \nabla^2 \log f_c(\boldsymbol{x})\right)\right)$, with $\mathbb{W}_{\boldsymbol{\Lambda}_c}$ denoting the law of a Brownian bridge $\{\boldsymbol{X}_t^{(c)}, t \in [0, T]\}$ with $\boldsymbol{X}_0^{(c)} := \boldsymbol{x}^{(c)}$, $\boldsymbol{X}_T^{(c)} := \boldsymbol{y}$ and covariance matrix $\boldsymbol{\Lambda}_c$

Scalability with sub-posterior correlation

In our *Generalised* Monte Carlo Fusion [Chan et al., 2021, Section 2] setting:

- Able to incorporate covariance / correlation information within our proposals and through p_c and h (in MCF $\mathbf{\Lambda}_c = \mathbb{I}_d$ for $c = 1, \ldots, C$)
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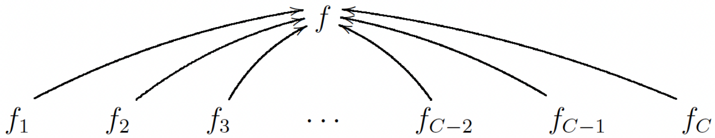
Problem: Scalability with C

The (Generalised) Monte Carlo Fusion algorithm implies a fork-and-join approach:



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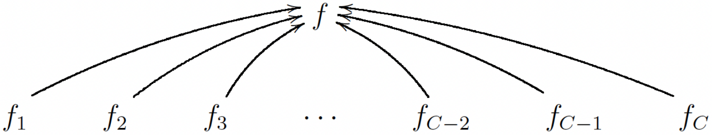

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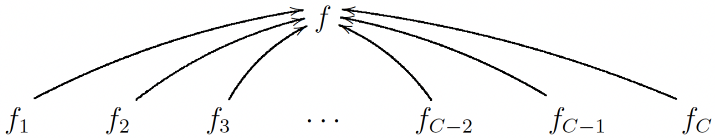

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- Solution: Divide-and-Conquer Monte Carlo Fusion [Chan et al., 2021, Section 3]
  - We could perform fusion in a proper divide-and-conquer framework
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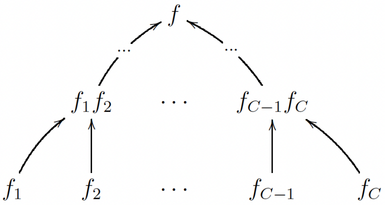
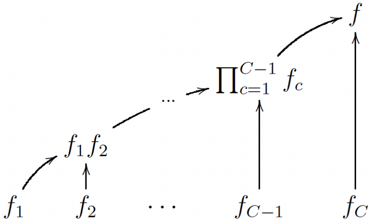
  - Two possible choices are balanced-binary (left) and progressive (right) trees





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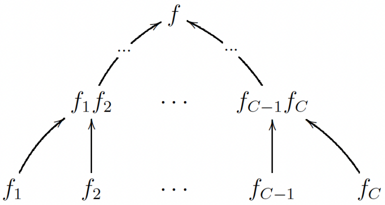

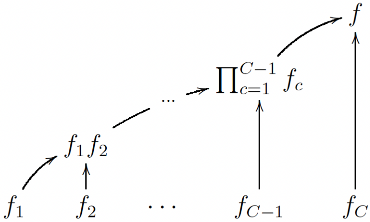

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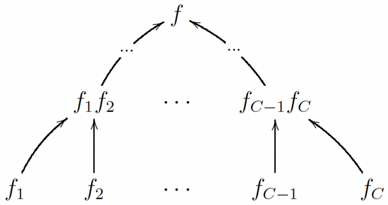

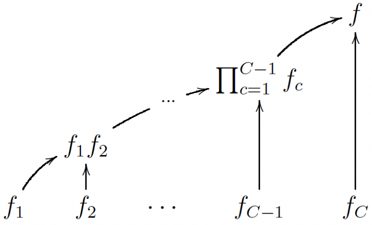

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Generalised Bayesian Fusion

Problem: Robustness to conflicting sub-posteriors

- Generalising the Bayesian Fusion approach of Dai et al. [2023]
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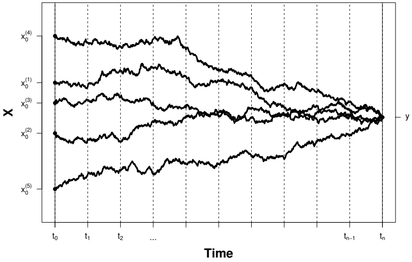
  - Want to make T large so that ρ_0 is large - but this makes ρ_1 smaller (since we have to simulate a diffusion over a longer time horizon T)
- Solution: Introduce temporal partition of T
  - Have the flexibility to choose T large enough for initialisation, while being able to have small intervals in the partition



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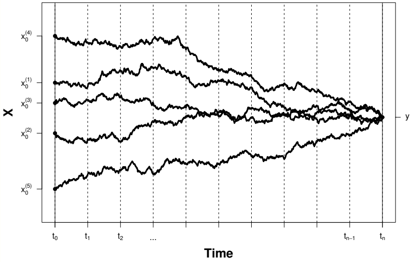

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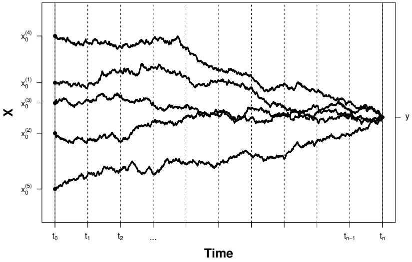

Examples

- We compare our methodology (Divide-and-Conquer Generalised Bayesian Fusion (D&C-GBF) [Chan et al., 2023]) with the approximate methodologies KDEMC [Neiswanger et al., 2014], WRS [Wang and Dunson, 2013] and CMC [Scott et al., 2016]
- To compare methods we calculate the integrated absolute distance metric

$$IAD = \frac{1}{2d} \sum_{j=1}^{d} \int \left| \hat{f}(\boldsymbol{x}_j) - f(\boldsymbol{x}_j) \right| \mathrm{d}\boldsymbol{x}_j \in [0,1]$$

where $\hat{f}(\boldsymbol{x}_j)$ is the marginal density for $\boldsymbol{x}_j$ based on the method applied (computed using a kernel density estimate) and $f(\boldsymbol{x}_j)$ is target marginal density

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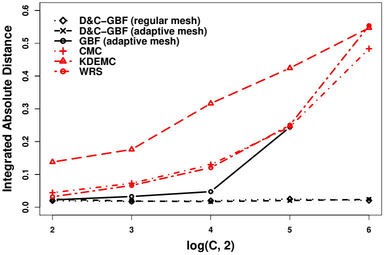
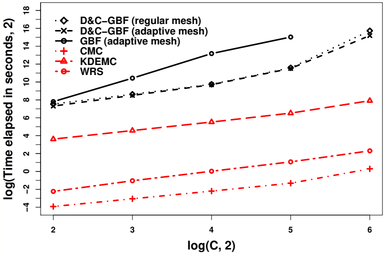
Logistic regression - simulated data

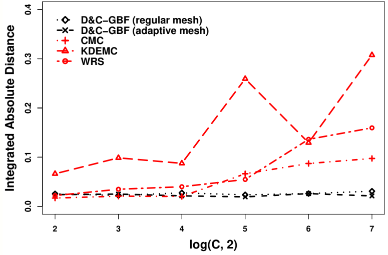
- Simulated data example with $n = 1000$ and $d = 5$
  - Small data size means that large data assumptions will fail
- We split the data into $C = 4, 8, 16, 32, 64$ and apply D&C-GBF (using a balanced binary tree approach)

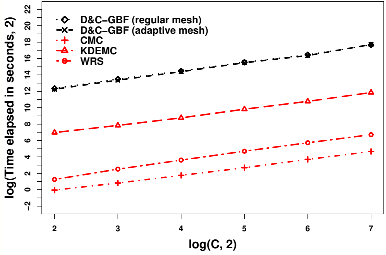




Logistic regression - nycflights13

- Applying logistic regression model to the nycflights13 dataset [Wickham, 2021] to predict the binary outcome of arrival-delay: $n = 327346$ and $d = 21$
- We split the data into $C = 4, 8, 16, 32, 64, 128$ and apply D&C-GBF (using a balanced binary tree approach)





Ongoing research questions

- Further reducing the computational cost of the Fusion approach
  - Exactness comes at a cost
- Practical implementation considerations for specific applications:
  - Big data setting: evaluating ϕ_c has $\mathcal{O}(m_c)$ cost - can perhaps employ sub-sampling methods to reduce this cost
  - Confidential fusion (Con-fusion): where sharing information/data between cores is not permitted
- Scalability with dimension
  - Performance with regards to dimension has improved since MCF, but not been explicitly addressed

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