Hierarchical Monte Carlo Fusion

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Outline

Fusion Problem

- Target:

$$\pi(\boldsymbol{x}) \propto \prod_{c=1}^{C} f_c(\boldsymbol{x})$$

  where each *sub-posterior*, $f_c(\boldsymbol{x})$, is a density representing one of the C distributed inferences we wish to unify
- Assume we can sample $\boldsymbol{x}^{(c)} \sim f_c(\boldsymbol{x})$
- Applications:
  - Expert elicitation: combining views of multiple experts
  - Privacy setting
  - Big Data (by construction)
  - Tempering (by construction)

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Fusion for Big Data

- Consider we have data $\boldsymbol{x}$ with a large number of observations n
- The likelihood $\ell(\boldsymbol{x} \mid \theta)$ becomes expensive to calculate
  - This makes MCMC prohibitively slow for big data
- Potential solution:

$$\pi(\theta \mid \boldsymbol{x}) \propto \prod_{i=1}^{n} \ell(\boldsymbol{x} \mid \theta)\pi(\theta) = \prod_{c=1}^{C} \ell(\boldsymbol{x}_c \mid \theta)\pi(\theta)^{\frac{1}{C}}$$

where $\boldsymbol{x}_c$ denotes the c-th subset for $c = 1, \ldots, C$ and $\pi(\theta) = \prod_{c=1}^{C} \pi(\theta)^{\frac{1}{C}}$ is the prior

- Advantage: inference on each smaller dataset can be conducted independently and in parellel

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Fusion for Tempering

- Consider the *power-tempered target distribution* $\pi_\beta(\boldsymbol{x}) = [\pi(\boldsymbol{x})]^\beta$ for $\beta \in (0, 1]$
- MCMC can become computationally expensive to sample from multi-modal densities and can get stuck in modes
- Potential solution:

$$\pi(\boldsymbol{x}) = \pi(\boldsymbol{x})^{\frac{1}{\beta} \cdot \beta} = \prod_{c=1}^{\frac{1}{\beta}} \pi_\beta(\boldsymbol{x})$$

where $\frac{1}{\beta} \in \mathbb{N}$

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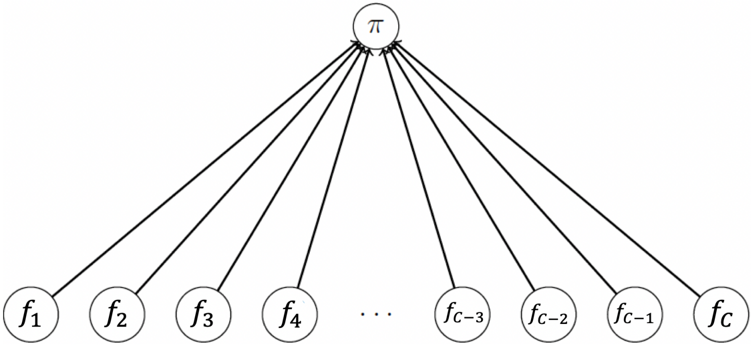
Fork-and-join

The fork-and-join approach:



Current Fork-and-Join Methods

- Several fork-and-join methods have been developed. For instance
  - Kernel density averaging [Neiswanger et al., 2013]
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Constructing a rejection sampler - An Extended Target

Proposition

Suppose that $p_c(\boldsymbol{y} \mid \boldsymbol{x}^{(c)})$ is the transition density of a stochastic process with stationary distribution $f_c^2(\boldsymbol{x})$. The $(C+1)d$-dimensional (fusion) density proportional to the integrable function

$$g(\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(C)}, \boldsymbol{y}) \propto \prod_{c=1}^{C} \left[f_c^2(\boldsymbol{x}^{(c)}) p_c(\boldsymbol{y} \mid \boldsymbol{x}^{(c)}) \cdot \frac{1}{f_c(\boldsymbol{y})} \right]$$

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Rejection Sampling (Double Langevin Approach)

- There are many possible choices for $p_c(\boldsymbol{y} \mid \boldsymbol{x})$
- Let $p_c(\boldsymbol{y} \mid \boldsymbol{x}) := p_{T,c}(\boldsymbol{y} \mid \boldsymbol{x})$, the transition density of the d-dimensional (double) Langevin (DL) diffusion processes $\boldsymbol{X}_t^{(c)}$ for $c = 1, \ldots, C$, from $\boldsymbol{x}$ to $\boldsymbol{y}$ for a pre-defined time $T > 0$ given by

$$\mathrm{d}\boldsymbol{X}_t^{(c)} = \nabla \log f_c(\boldsymbol{X}_t^{(c)})\mathrm{d}t + \mathrm{d}\boldsymbol{W}_t^{c},$$

  - $\boldsymbol{W}_t^{(c)}$ is d-dimensional Brownian motion
  - ∇ is the gradient operator over $\boldsymbol{x}$
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$$g(\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(C)}, \boldsymbol{y}) \propto \prod_{c=1}^{C} \left[f_c^2(\boldsymbol{x}^{(c)}) p_{T,c}(\boldsymbol{y} \mid \boldsymbol{x}^{(c)}) \cdot \frac{1}{f_c(\boldsymbol{y})} \right]$$

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$$h(\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(C)}, \boldsymbol{y}) \propto \prod_{c=1}^{C} \left[f_c\left(\boldsymbol{x}^{(c)}\right) \right] \cdot \exp\left(- \frac{C \cdot \|\boldsymbol{y} - \bar{\boldsymbol{x}}\|^2}{2T} \right)$$

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Rejection Sampling - acceptance probability

- Acceptance probability:

$$\frac{g(\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(C)}, \boldsymbol{y})}{h(\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(C)}, \boldsymbol{y})} \propto \rho \cdot Q$$

where

$$\begin{cases} \rho := e^{-\frac{C\sigma^2}{2T}}, \qquad \sigma^2 = \frac{1}{C}\sum_{c=1}^{C} \left\| \boldsymbol{x}^{(c)} - \bar{\boldsymbol{x}} \right\|^2 \\ Q := \mathbb{E}_{\bar{\mathbb{W}}}\left(\prod_{c=1}^{C} \left[\exp\left\{ -\int_0^T \left(\phi_c(\boldsymbol{x}_t^{(c)}) - \Phi_c \right) \mathrm{d}t \right\} \right] \right) \end{cases}$$

where $\bar{\mathbb{W}}$ denotes the law of C independent Brownian bridges $\boldsymbol{x}_t^{(1)}, \ldots, \boldsymbol{x}_t^{(C)}$ with $\boldsymbol{x}_0 = \boldsymbol{x}^{(c)}$ and $\boldsymbol{x}_T^{(c)} = \boldsymbol{y}$

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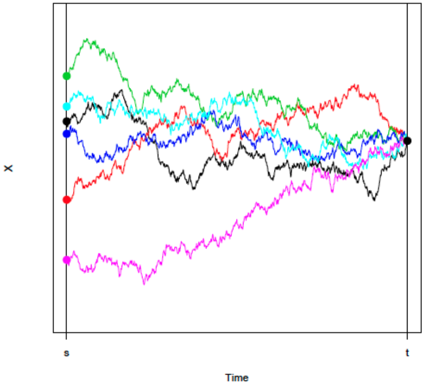
where

- $\phi_c(\boldsymbol{x}) = \frac{1}{2}\left(\|\nabla \log f_c(\boldsymbol{x})\|^2 + \Delta \log f_c(\boldsymbol{x})\right)$
- Φ_c are constants such that for all $\boldsymbol{x}$, $\phi_c(\boldsymbol{x}) \geq \Phi_c$ for $c \in \{1, \dots, C\}$
- Events of probability Q can be simulated using Poisson thinning and methodology called Path-space Rejection Sampling (PSRS) or the Exact Algorithm (Beskos et al. [2005], Beskos et al. [2006], Pollock et al. [2016])

Interpretation

- Correct a simple average $\bar{\mathbf{x}}$ of sub-posterior values to a Monte Carlo draw from $\pi(\mathbf{x})$ with acceptance probability $\rho \cdot Q$



Double Langevin Approach - Summary

- Proposal:

$$h(\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(C)}, \mathbf{y}) \propto \prod_{c=1}^{C} \left[f_c\left(\mathbf{x}^{(c)}\right) \right] \cdot \exp\left(- \frac{C \cdot \|\mathbf{y} - \bar{\mathbf{x}}\|^2}{2T} \right)$$

- Accept $\mathbf{y}$ as a draw from fusion density π with probability:

$$\frac{g(\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(C)}, \mathbf{y})}{h(\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(C)}, \mathbf{y})} \propto \rho \cdot Q$$

- Monte Carlo Fusion Algorithm:
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Limitations of above rejection sampler

1. Scalability: the acceptance probability of Monte Carlo fusion can be small, especially when C is large or d is large
2. Use of simple average $\bar{x}$ of sub-posterior samples as the proposal
   - but should we use a weighted average?
3. Use of same time T for each sub-posterior
   - but different cores / sub-posteriors could contain different amounts of information
4. PSRS: methodology for PSRS can be computationally expensive

Aim: To construct a fusion algorithm / framework to alleviate some of these limitations

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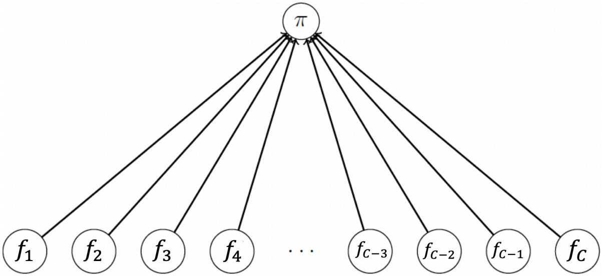
Hierarchical Monte Carlo Fusion

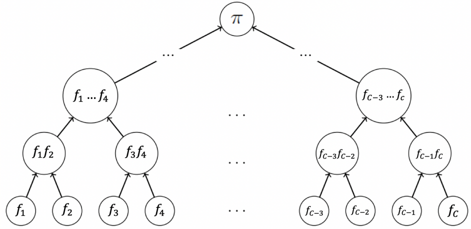
1. Scalability with C

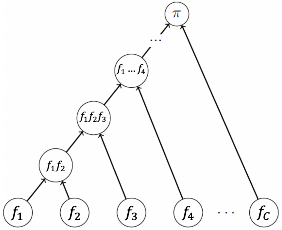
The Monte Carlo Fusion algorithm implies a fork-and-join approach:



Hierarchical Monte Carlo Fusion

- Solution: Hierarchical Monte Carlo Fusion
  - We could perform fusion in a proper divide-and-conquer framework
  - Two possible choices are hierarchical (left) and progressive (right) trees





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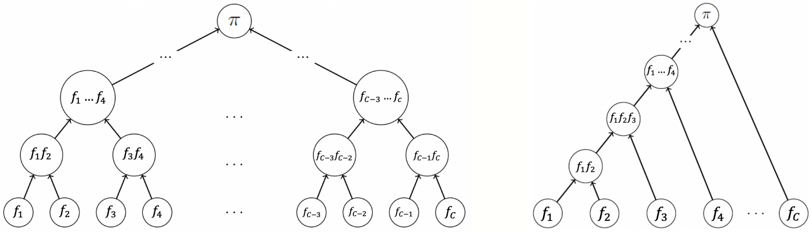

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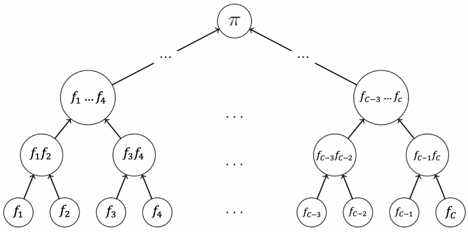

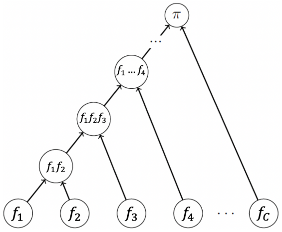

Note: Other trees are possible

Hierarchical Monte Carlo Fusion

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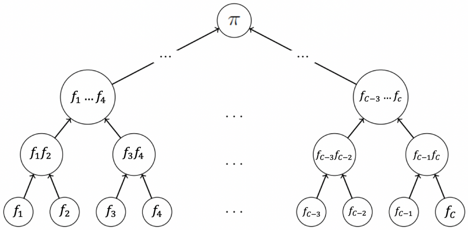
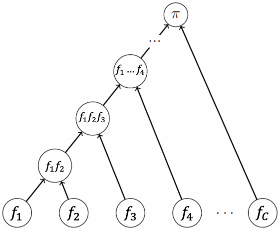
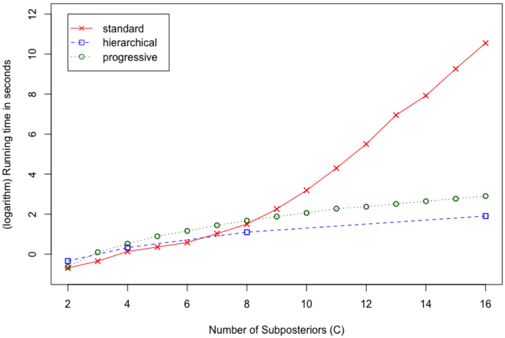




Note: Other trees are possible

Example

- Target: $\pi(x) \propto e^{-\frac{x^4}{2}}$
- Sub-posteriors: $f_c(x) = e^{-\frac{x^4}{2C}}$ for $c = 1, \ldots, C$

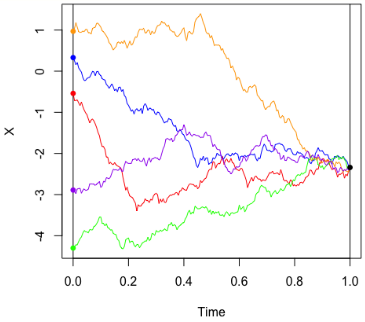


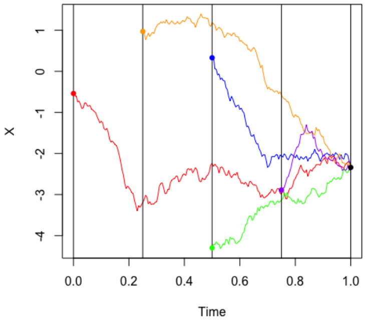
Time-adapting Monte Carlo Fusion

2. Use of simple average $\bar{\boldsymbol{x}}$ of sub-posterior samples as the proposal
3. Use of same time T for each sub-posterior

Time-adapting Monte Carlo Fusion

- Solution: Time-adapting Monte Carlo Fusion
  - We assign weights to each sub-posterior and use a weighted average: $\tilde{\mathbf{x}} = \sum_c w_c x^{(c)} / \sum_c w_c$
  - Time T is adapted for each posterior: $T_c = \frac{T}{w_c}$ for $c = 1, \ldots, C$





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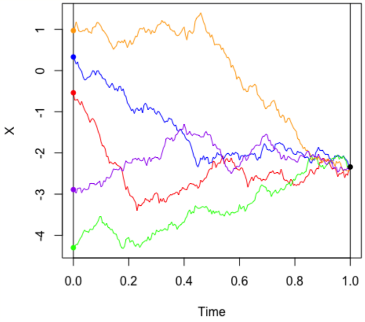

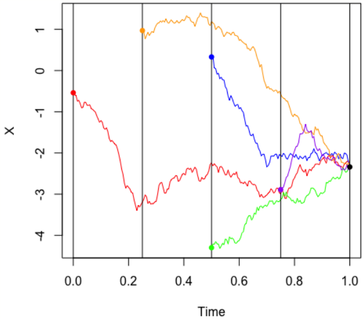

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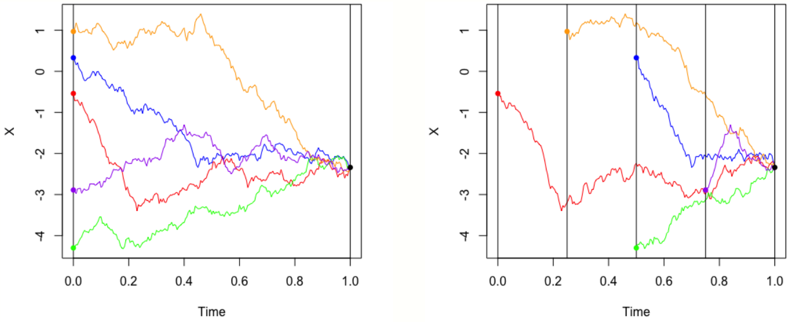
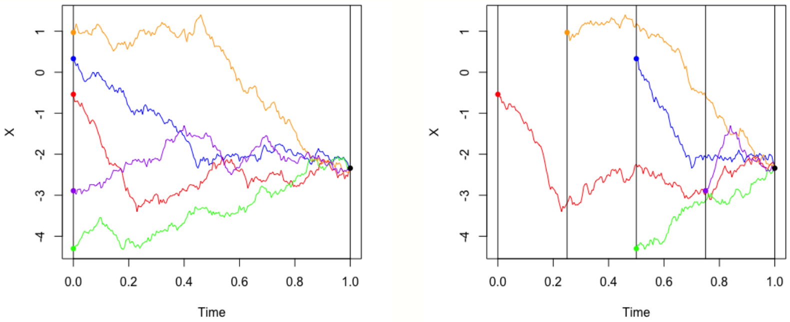


Time-adapting Monte Carlo Fusion

- Time-adapting Monte Carlo Fusion Algorithm:
  1. Choose time T and weights for sub-posteriors $w_c, c = 1, \ldots, C$
  2. Simulate $\boldsymbol{x}^{(c)} \sim f_c(\boldsymbol{x})$ and $\boldsymbol{y} \sim \mathcal{N}(\tilde{\boldsymbol{x}}, \frac{T\mathbb{I}_d}{\sum_c w_c})$
  3. Accept $\boldsymbol{y}$ with probability $\rho^{ta} \cdot Q^{ta}$



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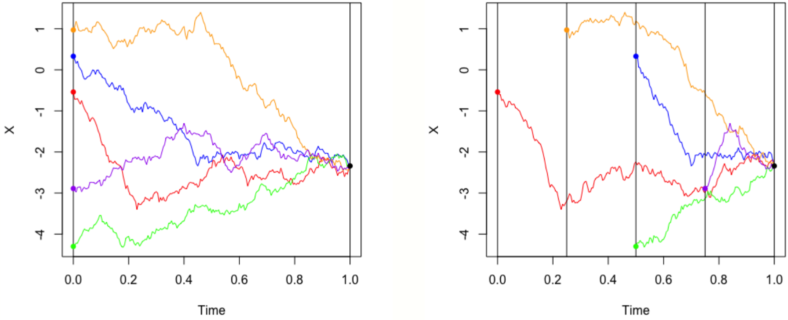

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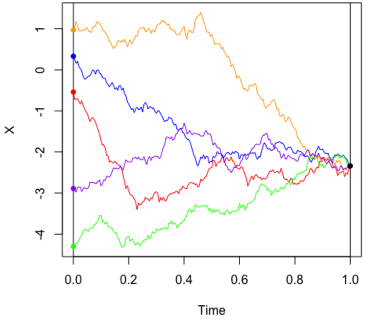

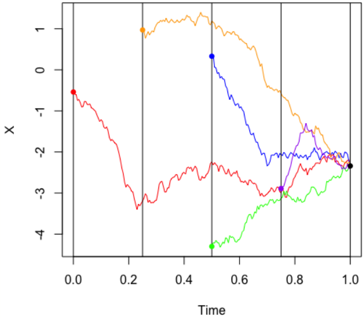

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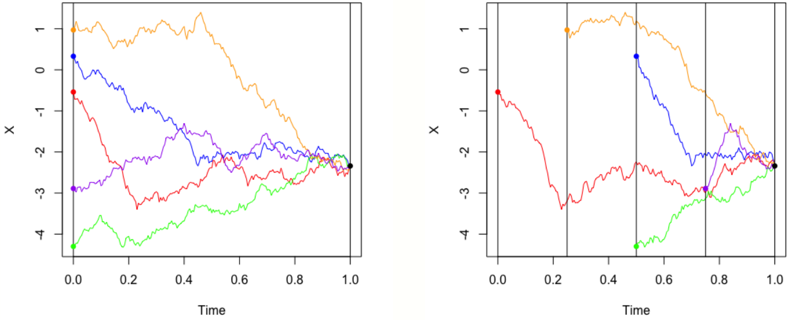

Divide-and-Conquer SMC

4. PSRS: methodology for PSRS can be computationally expensive

- Solution: Sequential Monte Carlo in the hierarchical fusion framework
- Rejection sampling can be wasteful: large number of proposed samples are rejected
- Motives the use of Sequential Importance Sampling / Resampling ideas
  - Replace the rejection sampling steps with importance sampling steps
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- Predicting if customers defaulted on their payments using gender and education levels ($n = 30,000$ and $d = 5$)
- Split into $C = 32$ subsets and apply the following methods:
    1. Consensus Monte Carlo [Scott et al., 2016]
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How to apply Hierarchical Time-adapting SMC Fusion

- Choose a time $T = 0.5$
- Use balanced hierarchical tree where we combine $m = 2$ sub-posteriors at each level, i.e. have $L = 6$ levels in the tree
- Weights are chosen according to how much data they have relative to the bottom level:
    - At $L = 6$: sub-posteriors are given weight $w_c = 1$ for $c = 1, \ldots, 32$, i.e. $T_c = 0.5$
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$$IAD = \frac{1}{2d} \sum_{j=1}^{d} \int \left| \hat{f}(\boldsymbol{x}_j) - f(\boldsymbol{x}_j) \right| \mathrm{d}x_j$$

  where $\hat{f}(\boldsymbol{x}_j)$ is the marginal density for $\boldsymbol{x}_j$ based on the method applied and $f(\boldsymbol{x}_j)$ is the benchmark estimate (obtained using NUTS with Stan)
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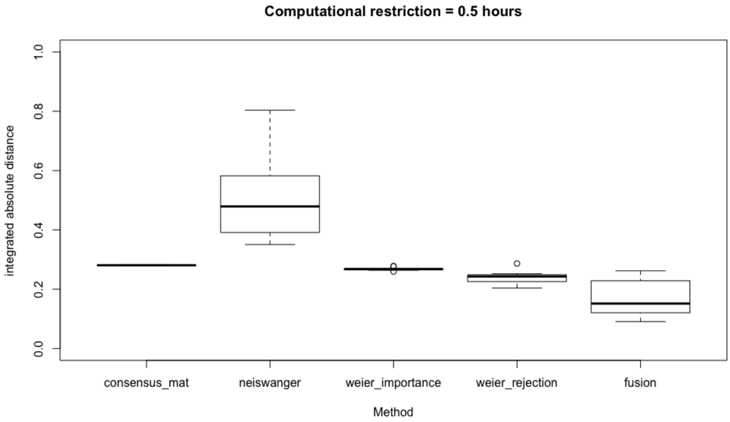

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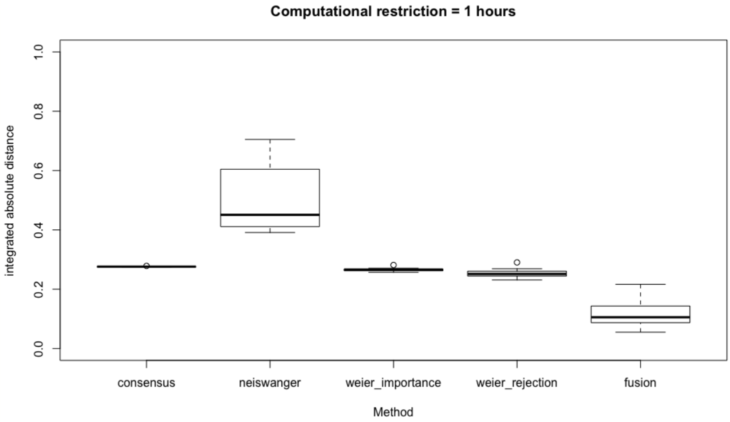

Ongoing directions

- Combining conflicting sub-posteriors
- Confidential fusion (Con-fusion): where sharing information/data between cores is not permitted
- Bayesian Fusion: tailored to big data Bayesian problems
- Different proposals: e.g. Ornstein-Uhlenbeck bridges, more general Langevin diffusion with pre-conditioning matrix for each sub-posterior
- Theory for D&C-SMC with Monte Carlo Fusion

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